

February 1995

# Modified tetrahedron equation and related 3D integrable models,II

H. E. Boos<sup>1</sup>

Institute for High Energy Physics,  
Protvino, Moscow Region, Russia

## Abstract

This work is a continuation of paper [13] where the Boltzmann weights for the N-state integrable spin model on the cubic lattice has been obtained only numerically. In this paper we present the analytical formulae for this model in a particular case. Here the Boltzmann weights depend on six free parameters including the elliptic modulus. The obtained solution allows to construct a two-parametric family of the commuting two-layer transfer matrices. Presented model is expected to be simpler for a further investigation in comparison with a more general model mentioned above.

---

<sup>1</sup>e-mail: boos@mx.ihep.su

# 1. Introduction

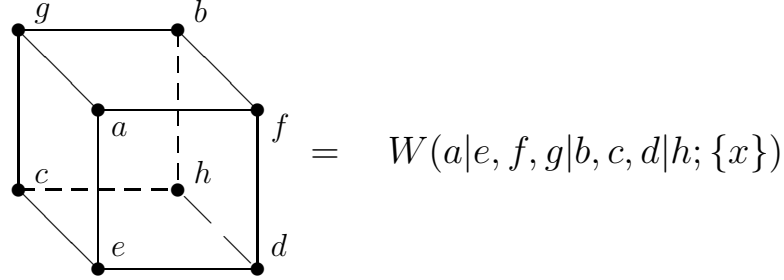
In paper [13] a new family of N-state integrable three dimensional two-layer models on cubic lattice has been formulated. The weight functions of this model have the composite structure. Namely, they consist of eight more elementary weights which depend on eight indices and four parameters  $x$ . These weights are a generalization used in [11] of the weights for Baxter- Bazhanov model [7]-[8]. They have the "Body-Centered-Cube" (BCC) structure firstly introduced by Baxter in [4] (more detailed [5]) for Zamolodchikov's model with N=2 ([1]-[2]). In papers [11]-[12] the idea to use a pair of so-called modified tetrahedron equations instead of a single tetrahedron one has been proposed. Paper [13] is a further generalization of this idea. Contrary to [11]-[12] the elementary weights should satisfy only one of two sets of modified tetrahedron equations. It allows to find a wider solution which provides the integrability of the above mentioned composite model. It may be parameterized by the elliptic functions which depend on the "angle-like" parameters and one more parameter  $k$  - the elliptic modulus. In these terms each elementary weight depends on three "angle-like" parameters and modulus  $k$  instead of the four parameters  $x$ . Any two weights corresponding to the neighbouring cubes in the lattice have one coinciding "angle-like" parameter and opposite elliptic modulae:  $k$  and  $-k$ . Rather cumbersome analysis has shown us that the solution of the tetrahedron equations for the composite model has twelve free parameters including elliptic modulus. This result has been obtained only numerically. We have no analytical formulae for this general case.

Our aim in this paper is to consider a simple particular case which could be more useful for a calculation of the statistical sum and other physical quantities. At the end of Section 6 of [13] an example of such a model has been considered but it seems to be trivial. In this paper we present another particular case which is probably more substantial.

The paper is organized as follows: In Section 2 we remind the reader of the basic formulae and notations used in papers [9]-[13]. In Section 3 we give all necessary constraints and their solution for one composite weight  $\mathcal{W}$ . Also, we mention one interesting property of this solution for our particular case:  $Z$ -invariance for the composite weights [6]. This property works only in two-layer level for the general case. In Section 4 we discuss the necessary conditions for the existence of the intertwining composite weights for  $\mathcal{W}$  and  $\mathcal{W}'$ . The resulting formulae are collected in the Appendix.

## 2. Formulation of the model

We would like to recall the basic notations used in [9]-[13]. First of all let us remind the reader that we shall consider a spin model on the cubic lattice. In the Baxter-Bazhanov case the Boltzmann weights corresponding to the cells of this lattice are the same. This model may be called homogeneous. Each Boltzmann weight depends on eight indices and parameters  $x$  (see Fig.1):



**Fig. 1**

Up to some face factors we have

$$W(a|e, f, g|b, c, d|h; \{x\}) = \left\{ \sum_{\sigma \in \mathbb{Z}_N} \frac{w(x_3, x_{13}, x_1|d, h + \sigma)w(x_4, x_{24}, x_2|a, g + \sigma)}{w(x_6, x_{56}, x_5|e, c + \sigma)w(x_8/\omega, x_{78}, x_7|f, b + \sigma)} \right\}_0 \quad (2.1)$$

where the lower index “0” after the curly brackets implies that the expression in the curly brackets is divided by itself with all exterior spin variables equated to zero. The parameters  $x_i$  and  $x_{ij}$  satisfy the Fermat condition:

$$x_{ij}^N = x_i^N - x_j^N. \quad (2.2)$$

In (2.1) we used the following notations:

$$\begin{aligned} w(x, y, z|k, l) &= w(x, y, z|k - l)\Phi(l), \\ w(x, y, z|k) &= \prod_{s=1}^k \frac{y}{z - x\omega^s}, \\ \Phi(l) &= \omega^{l(l+N)/2}, \\ \omega &= \exp(2\pi i/N) \end{aligned} \quad (2.3)$$

and

$$x^N + y^N = z^N, \quad (2.4)$$

$l$  and  $k$  are elements of  $Z_N$ .

Below we shall use the following normalization conditions:

$$x_3 = x_4 = x_6 = x_8 = 1. \quad (2.5)$$

In fact, each weight  $W(a|e, f, g|b, c, d|h)$  depends only on four continuous parameters  $x_1, x_2, x_5, x_7$  which in the case of the Baxter-Bazhanov model are connected with each other by the following constraint:

$$x_1 x_2 = x_5 x_7. \quad (2.6)$$

The weights functions of such form satisfy the tetrahedron equation ([9]-[10]):

$$\begin{aligned} & \sum_d W(a_4|c_2, c_1, c_3|b_1, b_3, b_2|d) W'(c_1|b_2, a_3, b_1|c_4, d, c_6|b_4) \\ & \times W''(b_1|d, c_4, c_3|a_2, b_3, b_4|c_5) W'''(d|b_2, b_4, b_3|c_5, c_2, c_6|a_1) \\ = & \sum_d W'''(b_1|c_1, c_4, c_3|a_2, a_4, a_3|d) W''(c_1|b_2, a_3, a_4|d, c_2, c_6|a_1) \\ & \times W'(a_4|c_2, d, c_3|a_2, b_3, a_1|c_5) W(d|a_1, a_3, a_2|c_4, c_5, c_6|b_4). \end{aligned} \quad (2.7)$$

where weights  $W', W'', W'''$  depend on their own sets of four parameters  $\{x'_i\}, \{x''_i\}, \{x'''_i\}$ , accordingly.

We write (2.7) more briefly in the shorthand notations:

$$W W' W'' W''' = W''' W'' W' W. \quad (2.8)$$

Below we shall not demand the condition (2.6). So, each Boltzmann weight depends on four independent variables  $x$ .

Authors of [11]-[12] have proposed to consider a pair of the tetrahedron-like equations instead of a single one (2.8):

$$W \overline{W}' W'' \overline{W}''' = W''' \overline{W}'' W' \overline{W} \quad (2.9)$$

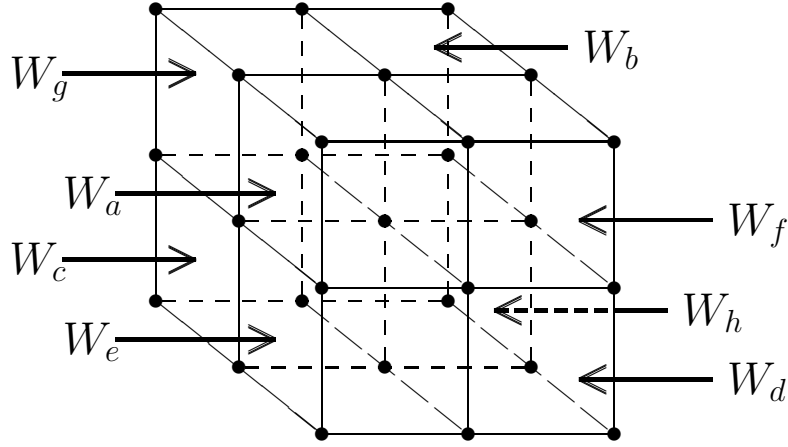
and

$$\overline{W} W' \overline{W}'' W''' = \overline{W}''' W'' \overline{W}' W \quad (2.10)$$

where  $\overline{W}$  depends on generally speaking some new variables  $\overline{x}_i$ . Equations of the form (2.9) or (2.10) are called modified tetrahedron equations. This idea allows the commutativity of two-layer transfer matrices for the cubic

lattice with a "chess" structure. The composite Boltzmann weight which corresponds to the  $2 \times 2 \times 2$  cube of this lattice satisfies the tetrahedron equation.

In our previous paper [13] we have generalized this idea in a following way. We have considered the composite weight  $\mathcal{W}$  which consists of eight generally speaking different elementary weights of the form (2.1) as shown in Fig.2:



**Fig. 2** A composite weight  $\mathcal{W}$ .

$$\mathcal{W} = \sum W(\{x_a\})W(\{x_b\})W(\{x_c\})W(\{x_d\}) \\ W(\{x_e\})W(\{x_f\})W(\{x_g\})W(\{x_h\}) \quad , \quad (2.11)$$

where sum  $\sum$  is implied to be over one internal index.

In order to satisfy the tetrahedron equation for such composite weights we have to consider sixteen modified tetrahedron equations for elementary weights of the form (ref.(6.3) of [13]):

$$W W' W'' W''' = \overline{W}''' \overline{W}'' \overline{W}' \overline{W}. \quad (2.12)$$

We would like to note that there is no such pair of equations among them which would be "conjugated" to each other as a pair of (2.9) and (2.10). If it was so we would be forced to restrict ourselves to the model proposed in [11], [12] . Unfortunately, this system of sixteen modified tetrahedron equations appears to be too complicated to be solved analytically. We have succeeded in

finding the solution to this system only numerically which depends on twelve free parameters. Also, a particular case of this model has been considered at the end of Section 6 of [13]. In particular, the opposite elementary weights within the composite one are connected with each other by the inversion for that case.

Now we shall consider another particular case with the following constraints on the opposite elementary weights:

$$W_a = \overline{W}_h, \quad W_b = \overline{W}_e, \quad W_c = \overline{W}_f, \quad W_d = \overline{W}_g. \quad (2.13)$$

The same relations will be implied to be valid for another composite weights  $\mathcal{W}', \mathcal{W}'', \mathcal{W}'''$ . Now the system of the sixteen modified tetrahedron equations mentioned above reduces to the following system of the eight ones:

$$\begin{aligned} W_g \overline{W}'_g W''_g \overline{W}'''_g &= W'''_g \overline{W}''_g W'_g \overline{W}_g, & \overline{W}_g W'_e \overline{W}''_e W'''_e &= \overline{W}'''_e W''_e \overline{W}'_e W_g, \\ W_e \overline{W}'_e W''_f \overline{W}'''_f &= W'''_f \overline{W}''_f W'_e \overline{W}_e, & \overline{W}_e W'_g \overline{W}''_h W'''_h &= \overline{W}'''_h W''_h \overline{W}'_g W_e, \\ W_f \overline{W}'_h W''_h \overline{W}'''_e &= W'''_e \overline{W}''_h W'_h \overline{W}_f, & \overline{W}_f W'_f \overline{W}''_f W'''_g &= \overline{W}'''_g W''_f \overline{W}'_f W_f, \\ W_h \overline{W}'_f W''_e \overline{W}'''_h &= W'''_h \overline{W}''_e W'_f \overline{W}_h, & \overline{W}_h W'_h \overline{W}''_g W'''_f &= \overline{W}'''_f W''_g \overline{W}'_h W_h. \end{aligned} \quad (2.14)$$

So, our problem has been reduced to finding the solution to this system for 32 elementary weights which depend on their own sets of four parameters  $x$  (see (2.1)). Below we shall consider this problem in more details.

It is more convenient to solve the system of modified equations (2.14) in a number of steps. First of all let us note that four modified equations among (2.14) have the form of (2.9) while four residual ones are of the form (2.10). So, our first step is to solve only one modified equation such as (2.9) or (2.10). Then we are going to resolve those constraints which connect with each other the parameters only of one composite weight  $\mathcal{W}$ . The same can be done for others  $\mathcal{W}', \mathcal{W}'', \mathcal{W}'''$ . The next step will be to find the necessary conditions on the parameters of the composite weights  $\mathcal{W}, \mathcal{W}'$  which provide the existence of the intertwining weights  $\mathcal{W}$  and  $\mathcal{W}'$ . Our last step reduces to applying the solution obtained in the first step to the modified equations (2.14) and to finding the elementary components for  $\mathcal{W}''$  and  $\mathcal{W}'''$ .

### 3. Solution of the modified tetrahedron equation

Let us consider the modified tetrahedron equation (2.9). We need some additional notations for one elementary weight  $W$  (2.1) which depends on the four parameters  $x_i$  as was mentioned above. Namely, let us introduce the following notations:

$$X_1 = x_1^N, \quad X_2 = x_2^N, \quad X_5 = x_5^N, \quad X_7 = x_7^N, \quad (3.1)$$

$$X_{13} = x_{13}^N, \quad X_{24} = x_{24}^N, \quad X_{56} = x_{56}^N, \quad X_{78} = x_{78}^N, \quad (3.2)$$

and taking into consideration conditions (2.5)

$$X_{13} = X_1 - 1, \quad X_{24} = X_2 - 1, \quad X_{56} = X_5 - 1, \quad X_{78} = X_7 - 1. \quad (3.3)$$

Then we can introduce a set of the four parameters  $\{m, J_i \ (i = 1, 2, 3)\}$  instead of  $X_i$ :

$$m^2 = \frac{X_1 X_2}{X_5 X_7}, \quad J_1 = \frac{X_7}{X_2}, \quad J_2 = \frac{X_5}{X_2}, \quad J_3 = \frac{X_{56} X_{78}}{X_{13} X_{24}}. \quad (3.4)$$

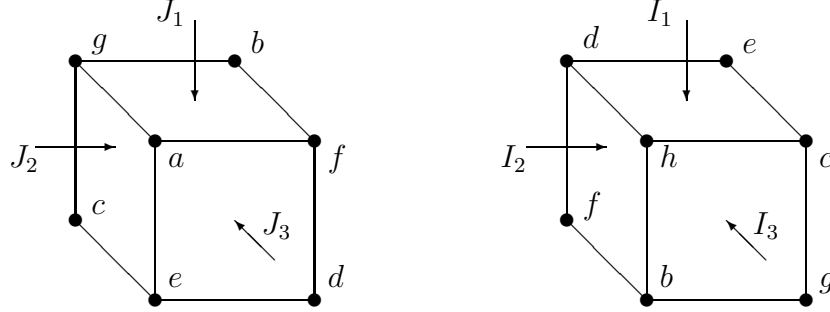
Note that  $m$  is connected with the elliptic modulus  $k$  introduced in [13] by the formula:

$$m = \frac{1 - k}{1 + k}$$

Also, it is convenient to introduce three more values

$$I_i = m^2 J_i. \quad (3.5)$$

These notations are slightly different from those which were used in [13]. These variables are different for the different weights. For the validity of the modified tetrahedron equations these variables should satisfy some definite relations which will be considered below. In order to make it more descriptive it is convenient to associate variables  $J_i$  with three cube's faces joined with a vertex "a" and  $I_i$  with three opposite faces joined with vertex "h" as shown in Fig.3.



**Fig. 3**

The connection with the "angle-like" variables  $\theta_i$  introduced in [13] is as follows:

$$J_i = \frac{1}{m} \left( \frac{cn(\theta_i, k) - dn(\theta_i, k)}{cn(\theta_i, k) + dn(\theta_i, k)} \right). \quad (3.6)$$

In [13], all algebraic relations which provide the validity of the modified tetrahedron equation (2.9) were done. One can choose from them those relations which contain the parameters of  $W, W', W'', W'''$  only:

$$\begin{aligned} J_1 &= J_2''', & J_2 &= J_2', & I_3 &= 1/J_2'', \\ I_1' &= 1/J_3''', & J_3' &= J_3'', & J_1'' &= J_1''' \end{aligned} \quad (3.7)$$

and

$$X_{24} X_{24}'' = X_{24}' X_{24}'''. \quad (3.8)$$

Below we shall call all relations similar to (3.8) as relations of type II. They connect together parameters of all weights. We are interested only in a particular case:

$$m = m' = m'' = m'''. \quad (3.9)$$

The analysis of the equations (3.7-3.9) is a bit cumbersome, but rather straightforward and can be easily done by *MATHEMATICA*. It can be shown that we have two different solutions. One of them seems to be meaningless and we shall discuss only the second one. Latter has six free parameters including the modulus. We will not write down the manifest formulae. We would like to note only that the variables  $X_i''$  and  $X_i'''$  of the weights  $W''$  and  $W'''$ , accordingly, can be expressed rationally through the variables  $X_i$



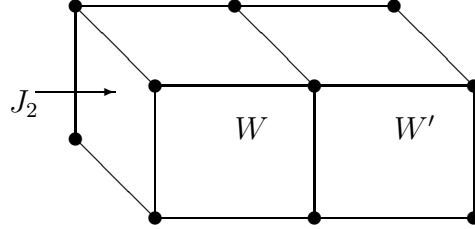
and  $X'_i$  of  $W$  and  $W'$ . The residual relations considered in [13] give us the variables for  $\overline{W}, \overline{W}', \overline{W}'', \overline{W}'''$ . Below we show them only for  $W$ , residual variables can be obtained analogically:

$$\begin{aligned}
\overline{m} &= 1/m, & (\overline{k} &= -k) \\
\overline{J}_i &= I_i, & \overline{I}_i &= J_i \\
\overline{X}_1 &= X_1 \frac{(J_3 m^2 - 1)}{m^2 (J_3 - 1)}, & \overline{X}_2 &= X_2 \frac{(J_3 m^2 - 1)}{m^2 (J_3 - 1)}, \\
\overline{X}_5 &= X_5 \frac{(J_3 m^2 - 1)}{(J_3 - 1)}, & \overline{X}_7 &= X_7 \frac{(J_3 m^2 - 1)}{(J_3 - 1)}.
\end{aligned} \tag{3.10}$$

We shall call the substitution  $W \rightarrow \overline{W}$  as  $T$ -transformation. If two weights  $W$  and  $W'$  correspond to the cubes which join each other by one face as it is shown in Fig.4, than we have

$$m' = \frac{1}{m}, \quad J'_2 = I_2, \quad I'_2 = J_2. \tag{3.11}$$

The relations of this kind we will call as relations of type I. They connect variables only of the neighbouring weights in the lattice.



**Fig. 4**

The equations (3.7) are relations of type I. They connect together the variables of two weights. Above we have only described how we can perform the first stage. Let us repeat that obtaining the manifest formulae is straightforward.

## 4. "Internal" constraints on the composite weight

In order to perform our following step we need some additional information. First of all, it is convenient for us to rewrite the equation (3.8) in a different form. Let us suppose that the equations (3.7) have already been satisfied. Then the equation of type II (3.8) is equivalent to the four relations of the "edge-like" parameters introduced in [13]:

$$\begin{aligned} a_0 + a'_2 - a''_2 + a'''_2 &= 0 \\ a_1 - a'_1 + a''_1 + a'''_0 &= 0 \\ a_2 + a'_0 - a''_3 + a'''_3 &= 0 \\ a_3 - a'_3 + a''_0 + a'''_1 &= 0, \end{aligned} \tag{4.1}$$

where  $a_i$  and  $a_0$  are determined by the following formulae:

$$\begin{aligned} \tan a_i/2 &= \sqrt{\frac{sn(\alpha_0, k)sn(\alpha_i, k)}{sn(\alpha_j, k)sn(\alpha_k, k)}}, \\ \tan a_0/2 &= k \sqrt{sn(\alpha_0, k)sn(\alpha_i, k)sn(\alpha_j, k)sn(\alpha_k, k)}, \end{aligned} \tag{4.2}$$

and  $\alpha_\mu$  are "excesses" :

$$\alpha_0 = \frac{\theta_1 + \theta_2 + \theta_3}{2} - \mathcal{K}, \quad \alpha_r = \theta_r - \alpha_0, \tag{4.3}$$

where  $\mathcal{K}$  is a complete elliptic integral of the first kind for the modulus  $k$ .

Below we also need the transformation properties of weights (2.1) obtained in [8],[9],[12] for the rotation  $\rho$  on  $\pi/2$  around the vertical axis and  $\tau$ -reflection. These two transformations are generating elements for a whole group of the cube's symmetry. Namely, weight (2.1) is invariant upon these transformations up to some face factors if variables  $X_i$  are changed by  $X_i^\rho$  for  $\rho$ -rotation and  $X_i^\tau$  for  $\tau$ -reflection. Also, we need  $X_i^\lambda$  for the rotation  $\lambda$  on  $2\pi/3$  around the a-h direction of the cube (Fig.2):

$$\rho : \{a, e, f, g, b, c, d, h\} \rightarrow \{g, c, a, b, f, h, e, d\}$$

$$X_1^\rho = \frac{X_{56}}{X_{13}}, \quad X_2^\rho = \frac{X_7 X_{24}}{X_2 X_{78}}, \quad X_5^\rho = \frac{X_1 X_{56}}{X_5 X_{13}}, \quad X_7^\rho = \frac{X_{24}}{X_{78}};$$

$$\tau : \{a, e, f, g, b, c, d, h\} \rightarrow \{a, f, e, g, c, b, d, h\}$$

$$X_1^\tau = X_1, \quad X_2^\tau = X_2, \quad X_5^\tau = X_7, \quad X_7^\tau = X_5;$$

$$\lambda : \{a, e, f, g, b, c, d, h\} \rightarrow \{a, g, e, f, d, b, c, h\}$$

$$\begin{aligned} X_1^\lambda &= \frac{X_1}{X_5} \frac{X_{78}}{X_{24}} \frac{(X_5 - X_2)}{(X_1 - X_7)}, & X_2^\lambda &= \frac{X_{13}}{X_{56}} \frac{(X_5 - X_2)}{(X_1 - X_7)}, \\ X_5^\lambda &= \frac{X_7}{X_2} \frac{X_{13}}{X_{56}} \frac{(X_5 - X_2)}{(X_1 - X_7)}, & X_7^\lambda &= \frac{X_{78}}{X_{24}} \frac{(X_5 - X_2)}{(X_1 - X_7)}. \end{aligned} \quad (4.4)$$

Now let us consider the eight modified tetrahedron equations (2.14). The analysis of the previous Section may be applied for each of them. So, we can write all relations of type I such as (3.7) and all relations of type II such as (4.1). Combining together these relations one can obtain those equations which are pure "internal" constraints on the variables of composite weights.

$$m_e = m_f = m_g = m_h = m,$$

$$\begin{aligned} J_{g1} &= J_{f1}, & J_{g2} &= J_{e2}, & J_{g3} &= J_{h3}, \\ J_{e1} &= J_{h1}, & J_{f2} &= J_{h2}, & J_{e3} &= J_{f3} \end{aligned} \quad (4.5)$$

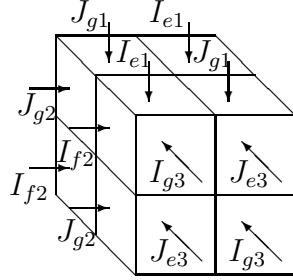
and

$$\begin{aligned} g_0 &+ e_0 + f_0 + h_0 = 0 \\ g_1 &+ f_1 = e_1 + h_1 \\ g_2 &+ e_2 = f_2 + h_2 \\ g_3 &+ h_3 = e_3 + f_3. \end{aligned} \quad (4.6)$$

where  $\{J_{ei}, I_{ei}, e_\mu\}$ ,  $\{J_{fi}, I_{fi}, f_\mu\}$ ,  $\{J_{gi}, I_{gi}, g_\mu\}$  and  $\{J_{hi}, I_{hi}, h_\mu\}$  are variables  $J_i, I_i$  defined by (3.4-3.5) and "edge-like" parameters defined by (4.2) for weights  $W_e, W_f, W_g$  and  $W_h$  accordingly. The analogical relations to (4.5-4.6) should be valid for other composite weights  $\mathcal{W}', \mathcal{W}'', \mathcal{W}'''$ . This situation is shown in Fig.5. The last four relations of type II may be replaced by four

relations of variables  $X_e, X_f, X_g, X_h$  for weights  $W_e, W_f, W_g, W_h$  accordingly:

$$\begin{aligned} \frac{X_{e13}}{\bar{X}_{e24}} \frac{X_{f24}}{\bar{X}_{f13}} \frac{X_{g78}}{\bar{X}_{g56}} \frac{X_{h56}}{\bar{X}_{h78}} &= \frac{X_{e24}}{\bar{X}_{e13}} \frac{X_{f13}}{\bar{X}_{f24}} \frac{X_{g56}}{\bar{X}_{g78}} \frac{X_{h78}}{\bar{X}_{h56}} = \\ &= \frac{X_{e56}}{\bar{X}_{e78}} \frac{X_{f78}}{\bar{X}_{f56}} \frac{X_{g24}}{\bar{X}_{g13}} \frac{X_{h13}}{\bar{X}_{h24}} = \frac{X_{e78}}{\bar{X}_{e56}} \frac{X_{f56}}{\bar{X}_{f78}} \frac{X_{g13}}{\bar{X}_{g24}} \frac{X_{h24}}{\bar{X}_{h13}} = \frac{X_{g1}}{m^2 \bar{X}_{g1}} \frac{X_{e1}}{\bar{X}_{e1}}. \end{aligned} \quad (4.7)$$



**Fig. 5**

The consequence of equations (4.5,4.7) are the following relations which appear to be very useful:

$$\begin{aligned} X_{f1} &= p X_{g1} \frac{X_{e1}}{\bar{X}_{e1}}, & \bar{X}_{f1} &= p X_{g1}, \\ X_{f2} &= q \frac{X_{e1}}{\bar{X}_{e1}}, & \bar{X}_{f2} &= q, \\ X_{f5} &= p X_{g5} \frac{X_{e1}}{\bar{X}_{e1}}, & \bar{X}_{f5} &= m^2 p X_{g5}, \\ X_{f7} &= q \frac{X_{g1}}{m^2 X_{g5}} \frac{X_{e1}}{\bar{X}_{e1}}, & \bar{X}_{f7} &= q \frac{X_{g1}}{X_{g5}}, \\ X_{h1} &= p X_{e1} \frac{X_{g1}}{\bar{X}_{g1}}, & \bar{X}_{h1} &= p X_{e1}, \\ X_{h2} &= q \frac{X_{e2}}{X_{g2}} \frac{X_{g1}}{\bar{X}_{g1}}, & \bar{X}_{h2} &= q \frac{X_{e2}}{X_{g2}}, \\ X_{h5} &= p X_{e2} \frac{X_{g5}}{X_{g2}} \frac{X_{g1}}{\bar{X}_{g1}}, & \bar{X}_{h5} &= p m^2 X_{e2} \frac{X_{g5}}{X_{g2}}, \\ X_{h7} &= \frac{q}{m^2} \frac{X_{e1}}{X_{g5}} \frac{X_{g1}}{\bar{X}_{g1}}, & \bar{X}_{h7} &= q \frac{X_{e1}}{X_{g5}}, \end{aligned} \quad (4.8)$$

where  $p$  and  $q$  are some new parameters. One can see that substitution of these expressions into equations (4.5) gives us a rational constraint on  $p$  and

q. After one of the relations (4.7) has been satisfied we have two different solutions for p and consequently q. For one of these solutions all three residual relations (4.7) are valid automatically and we have a six-parametric solution. For another one, three residual relations (4.7) give us one additional quadratic constraint and we have five parametric solution. We should note that this solution contains the Baxter-Bazhanov [7]-[8] model and Zamolodchikov's one [1]-[2] for  $N = 2$  as a particular case. Unfortunately, we have not succeeded in achieving a final result in that case. Below we will consider only the first case which is much easier. In this case the following expressions for p and q can be obtained:

$$p = \frac{(X_{e1}^\rho - X_{g5}^\lambda)}{(X_{e5}^\rho - X_{g2}^\lambda)} \frac{X_{g2}}{X_{e2}X_{g1}X_{g5}}, \quad q = \frac{(X_{e7}^\rho - X_{g1}^\lambda)}{(X_{e2}^\rho - X_{g7}^\lambda)} \frac{X_{g5}}{m^2 X_{e5}X_{g7}}. \quad (4.9)$$

where  $X_{ei}^\rho$  and  $X_{gi}^\lambda$  can be defined by applying the formulae (4.4) to  $W_e$  and  $W_g$ . It is easy to obtain a manifest formulae for variables  $X_{fi}, \overline{X}_{fi}$  and  $X_{hi}, \overline{X}_{hi}$  through the six independent variables (for example,  $X_{g1}, X_{g2}, X_{g5}, X_{e1}, X_{e2}$  and  $m$ ) using (3.10), (4.4), (4.8) and (4.9).

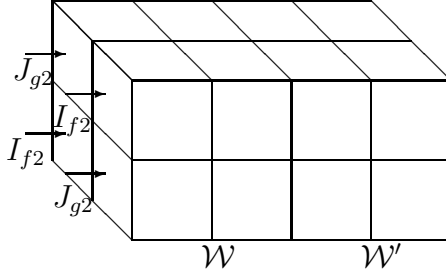
To conclude this Section we would like to mention a remarkable fact concerning our solution. Namely, let us consider two neighbouring weights within the composite weight  $\mathcal{W}$  (see Fig.1), for example,  $W_e$  and  $W_d$  which is now equal to  $\overline{W}_g$ . We can find the intertwining weights  $W''$  and  $W'''$  for this pair. In order to do so we should substitute  $X_{ei}$  and  $X_{gi}$  into the equations (4.4), (3.7) instead of  $X_i$  and  $X'_i$  accordingly and resolve them with respect to  $X''_i$  and  $X'''_i$ , as it was described in the previous Section. We can follow the same procedure for pairs  $W_a = \overline{W}_h$  and  $W_f$ ,  $W_c = \overline{W}_f$  and  $W_h$ ,  $W_g$  and  $W_b = \overline{W}_e$  and obtain the  $Y''_i$  and  $Y'''_i$ ,  $Z''_i$  and  $Z'''_i$ ,  $V''_i$  and  $V'''_i$  accordingly. It is interesting to note the following relations :

$$\begin{aligned} Z''_i &= \overline{X}''_i, & Y'''_i &= \overline{X}'''_i, \\ Y''_i &= \overline{V}''_i, & Z'''_i &= \overline{V}'''_i. \end{aligned} \quad (4.10)$$

One can check that the similar situation takes place for all pairs of the neighbouring elementary weights inside the composite weight. This property is nothing else but  $Z$ -invariance generalized on the three dimensional case ([6]). One should note that this property works for a general model (beyond the condition (2.13)) only in the two-layer level and breaks within the composite weight. We did not demand the validity of  $Z$ -invariance for the model (2.13). That is why we were very surprised to observe it.

## 5. Existence of the intertwining weights

Let us consider the question about the existence of the intertwining composite weights  $\mathcal{W}''$  and  $\mathcal{W}'''$  for two original weights  $\mathcal{W}$  and  $\mathcal{W}'$  which can be associated with the two cubes joining each other by the face as shown in Fig.6:



**Fig. 6**

As was mentioned above each of the weights  $\mathcal{W}$  and  $\mathcal{W}'$  has six free parameters (including modulus). Besides, we have three relations of type I:

$$m = m', \quad J_{g2} = J'_{g2}, \quad J_{f2} = J'_{f2}. \quad (5.1)$$

It seems to be natural that there are no another constraints on the parameters of  $\mathcal{W}$  and  $\mathcal{W}'$ . If it was so we would have three parametric family of the commuting two-layer transfer matrices. But accurate analysis has shown us that it is not the case. There is one more constraint which is necessary for the existence of  $\mathcal{W}''$  and  $\mathcal{W}'''$ . So, the commuting family is only two- parametric.

After the analysis of the previous Section has been performed it is enough to resolve the following set of twelve equations of type I:

$$\begin{aligned} J_{g1} &= J_{g2}''' , & J_{g2} &= J'_{g2}, & J_{g3} &= 1/I_{g2}'', \\ J_{g1}' &= 1/I_{g3}''' , & J'_{g3} &= J''_{g3}, & J''_{g1} &= J'''_{g1}, \\ J_{e1} &= J_{f2}'' , & J_{e3} &= 1/I_{f2}'', & J'_{e1} &= 1/I_{e3}''', \\ J'_{e3} &= J''_{e3} , & J_{f2} &= J'_{f2}, & J''_{e1} &= J'''_{e1} \end{aligned} \quad (5.2)$$

and four relations of type II:

$$X_{g24} X_{g24}'' = X_{g24}' X_{g24}''',$$

$$\begin{aligned}
X_{e24} X_{f24}'' &= X_{e24}' X_{f24}''', \\
X_{f24} X_{h24}'' &= X_{h24}' X_{e24}''', \\
X_{h24} X_{e24}'' &= X_{f24}' X_{h24}'''.
\end{aligned} \tag{5.3}$$

Below we describe our final result. Our choice of independent variables is as follows:

modulus  $m$ , five parameters from  $\mathcal{W}$ :  $X_{g1}, X_{g2}, X_{g5}, X_{e1}, X_{e2}$  and two parameters from  $\mathcal{W}'$ :  $X'_{g1}, X'_{g2}$ . In order not to encumber the text we have collected the resulting formulae for  $X'_{ei}$  in the Appendix.

Variables  $\overline{X}'_{ei}$  can be obtained from the formulae (3.10) and definitions (3.4) applied to weight  $W'_e$ . Expressions for  $X'_{fi}, \overline{X}'_{fi}$  and  $X'_{hi}, \overline{X}'_{hi}$  can be extracted by substituting already known values into the formulae (4.8) for the composite weight  $\mathcal{W}'$ . Also, it is necessary to use the following expressions for  $p'$  and  $q'$ :

$$p' = \frac{p}{S_1}, \quad q' = \frac{q}{S_1} \frac{X'_{g2}}{X_{g2}}. \tag{5.4}$$

Now we know all variables for weights  $\mathcal{W}$  and  $\mathcal{W}'$  and use the analysis of Section 3 for finding all intertwining elementary weights from (2.14) using a known solution of the equations (3.7-3.8). It is remarkable to note that all "internal" constraints described in Section 4 are satisfied automatically. Let us remind the reader that we have solved all necessary equations for  $N$ 'th powers of the original variables  $x$ . Now we need to choose the right powers of  $\omega$  when extracting the  $N$ 'th roots. We have done it for a general situation and a particular case considered in Section 6 of [13]. This experience tells us that it can be done. Moreover, as a rule there is some arbitrariness in this procedure.

## 6. Conclusion

In this paper we have considered a particular case of the general two-layer integrable model proposed in [13]. In fact, we have discussed only one out of two possible solutions. This solution has five free parameters and one modulus which is the same for all composite weights. It is interesting to note that for this solution the so-called  $Z$ -invariance has been restored inside each composite weight while for the general model this property takes place only on the two-layer level. Finding of the intertwiners for a pair of the composite

weights  $\mathcal{W}$  and  $\mathcal{W}'$  appeared to be unexpectedly difficult. In spite of our expectation the commuting family of the transfer matrices has only two free parameters (not three). Unfortunately, we have not clarified completely the meaning of an additional constraint. The model discussed above is simple enough and at the same time it has enough free parameters. So, we hope that this model may be useful for study of it's thermodynamic properties.

### Acknowledgement

Author thanks Professor W. Zimmermann for his hospitality at the Max-Planck-Institut für Physik in München where author was enabled to discuss this paper. This work is supported by the International Scientific Fund (INTAS), Grant No. RMM000. Author would like to thank Yu.G. Stroganov for reading the paper and making many useful remarks. Also, author would like to thank V.V.Mangazeev, S.M.Sergeev, E. Seiler, M. Niedermaier, O. Ogievetsky, R. Flume and V. Rittenberg for very fruitful discussions. Computer calculations were carried out using *Mathematica*.

## 7. Appendix

Here we present the resulting formulae for  $X'_{ei}$ . These formulae being written in terms of the independent variables are rather cumbersome. To simplify them we need more notations:

$$\begin{aligned}
y_g &= X_{g5} X_{g13} X_{g24} (1 - I_{g3}), \\
z_g &= m^2 X_{g5} X_{g13} X_{g24} (1 - J_{g3}), \\
y_e &= X_{g2} X_{g5} X_{e13} X_{e24} (1 - I_{e3}), \\
z_e &= m^2 X_{g2} X_{g5} X_{e13} X_{e24} (1 - J_{e3}), \\
y'_g &= X_{g2} X_{g5} X'_{g13} X'_{g24} (1 - I'_{g3}), \\
z'_g &= m^2 X_{g2} X_{g5} X'_{g13} X'_{g24} (1 - J'_{g3}).
\end{aligned} \tag{7.1}$$

Note that

$$\frac{\overline{X}_{g1}}{X_{g1}} = \frac{y_g}{z_g} \quad \frac{\overline{X}_{e1}}{X_{e1}} = \frac{y_e}{z_e} \quad \frac{\overline{X}'_{g1}}{X'_{g1}} = \frac{y'_g}{z'_g}. \tag{7.2}$$

Also, let us introduce :

$$\begin{aligned}
\alpha &= m^2 X_{g5} y_g - X_{e1} z_g \\
\beta &= X_{g5} z_e - X_{e1} y_e
\end{aligned} \tag{7.3}$$



and

$$\begin{aligned}
w_1 &= X'_{g1} z_g - m^2 X_{g5} y_g, \\
w_2 &= X_{e1} X_{g1} X_{g2} - m^2 X_{e2} X_{g5}^2, \\
w_3 &= m^2 X_{g5} y'_g - X_{g1} z'_g, \\
w_4 &= X_{e1} X_{g2} y'_g - X_{g5} X_{e2} z'_g, \\
w_5 &= X_{e24} (X_{g2} - m^2 X_{g5}) (X_{g2} - X_{e2} X_{g5}) w_2 + \\
&\quad + (X'_{g2} - X_{e2}) X_{g5} (X_{g1} X_{g2} z_e - m^2 X_{e2} X_{g5} y_e). \tag{7.4}
\end{aligned}$$

After that our result looks as follows:

$$\begin{aligned}
X'_{e1} &= S_1 S_2, \quad X'_{e2} = X_{g2} S_1 S_3, \\
X'_{e5} &= \frac{X_{g5}}{X_{g2}} X'_{e2}, \quad X'_{e7} = \frac{X_{g2}}{m^2 X_{g5}} X'_{e1}, \tag{7.5}
\end{aligned}$$

where

$$S_1 = \frac{l_1}{r_1}, \quad S_2 = \frac{l_2}{r_2}, \quad S_3 = \frac{l_3}{r_3} \tag{7.6}$$

and

$$\begin{aligned}
l_1 &= X'_{g1} (X'_{g2} - X_{g2}) y_e \alpha - X'_{g2} w_1 \beta, \\
r_1 &= X_{g5} (X'_{g2} - X_{g2}) z_e \alpha - X_{g2} w_1 \beta, \\
l_2 &= w_1 w_2 y'_g - X_{e2} X_{g5} w_3 \alpha, \\
r_2 &= X_{g2} (X_{e1} - X'_{g1}) w_3 y_g + w_1 w_4, \\
l_3 &= X_{e2} X_{g2} w_3 \beta - (X'_{g2} - X_{g2}) w_2 z_e y'_g, \\
r_3 &= (m^2 - 1) X_{g2} X_{g5} (X_{e1} - X'_{g1}) (X'_{g2} - X_{g2}) w_5 + \\
&\quad + X_{g2} (X'_{g2} - X_{e2}) r_1 + l_3. \tag{7.7}
\end{aligned}$$

## References

- [1] A.B. Zamolodchikov, Zh. Eksp. Teor. Fiz. **79** (1980) 641-664 [English trans.: JETP **52** (1980) 325-336].
- [2] A.B. Zamolodchikov, Commun. Math. Phys. **79** (1981) 489-505.
- [3] R.J. Baxter, Commun. Math. Phys. **88** (1983) 185-205.

- [4] R.J. Baxter, Phys. Rev. Lett., Vol. 53,**19** (1984) 1795-1798.
- [5] R. J. Baxter, Physica D, 8 (1986) 321 – 347.
- [6] R.J. Baxter, Phys. Trans. Roy. Soc. (London),Vol.289 A, **1359** (1978) 315 - 346.
- [7] V.V. Bazhanov, R.J. Baxter, J. Stat. Phys. **69** (1992) 453-485.
- [8] V.V. Bazhanov, R.J. Baxter, J. Stat. Phys. **71** (1993) 839.
- [9] R.M. Kashaev, V.V. Mangazeev, Yu.G. Stroganov, Int. J. Mod. Phys. **A8** (1993) 1399-1409.
- [10] R.M. Kashaev, V.V. Mangazeev, Yu.G. Stroganov, Int. J. Mod. Phys. **A8** (1993) 587-601.
- [11] V.V. Mangazeev, Yu.G. Stroganov: *Elliptic solution for modified tetrahedron equations*. Preprint IHEP 93-80, 1993, (HEP-TH/9305145), to appear in Mod. Phys. Lett. A.
- [12] V.V. Mangazeev, S.M. Sergeev, Yu.G. Stroganov, to appear in Int. J. Mod. Phys.
- [13] H.E. Boos, V.V. Mangazeev, S.M. Sergeev, *Modified tetrahedron equation and related 3D integrable models*. Preprint IHEP 94-76, HEP-TH/9407146, to appear in Int. J. Mod. Phys..